15. MODULUS OF ELASTICITY

The modulus of elasticity (= Young's modulus) E is a material property, that describes its stiffness and is therefore one of the most important properties of solid materials.

Mechanical deformation puts energy into a material. The energy is stored elastically or dissipated plastically. The way a material stores this energy is summarized in stress-strain curves. Stress is defined as force per unit area and strain as elongation or contraction per unit length

When a material deforms elastically, the amount of deformation likewise depends on the size of the material, but the strain for a given stress is always the same and the two are related by Hooke´s Law (stress is directly proportional to strain):

$$\sigma = E \cdot \varepsilon$$

where σ is   stress [MPa]

E modulus of elasticity [MPa]

ε strain [unitless or %]

From the Hook's law the modulus of elasticity is defined as the ratio of the stress to the strain :

$$E = \frac{\sigma}{\varepsilon} \quad [\text{MPa}]$$

Stress is not directly measurable. We can calculate it from different formulas for different types of the loading (tension, flexural stress,…)

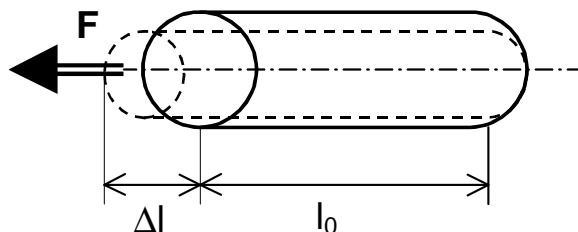Strain is defined as the change of the length divided by the original (initial) length (see Fig.:38) :

$$\varepsilon = \frac{\Delta l}{l_0} = \frac{l_1 - l_0}{l_0} \quad [\text{unitless or } \%]$$

where Δl is   change of the length   [m]

l_1 length after elongation [m]

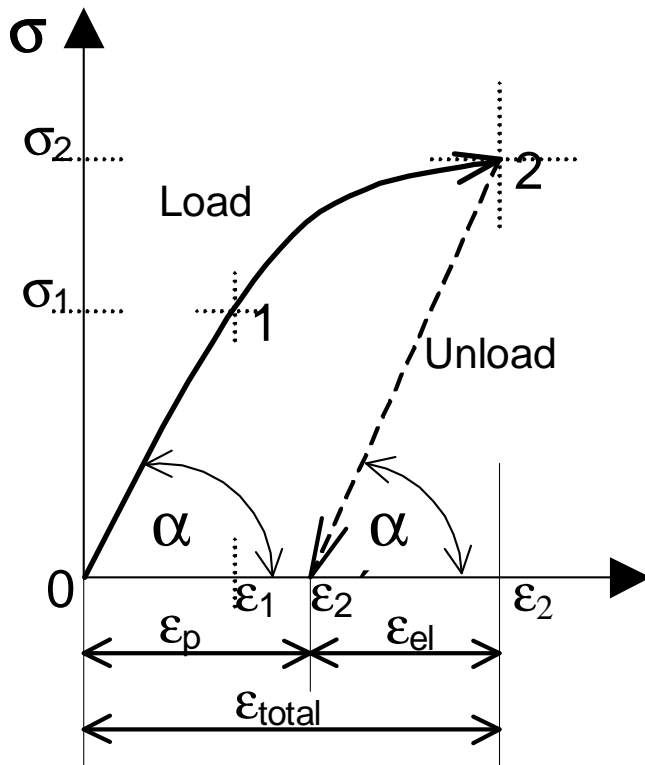l_0 original (initial) length [m]

Fig.:38 Determination of the strain

15.1 Graphical Determination of Modulus of Elasticity

Graphically we can define modulus of elasticity as a slope of the linear portion of the stress-strain diagram (see Fig.:39)

Fig.:39 Graphical relationship between total strain, permanent strain and elastic strain

$$E = tg\ \alpha$$

ε_p ......permanent strain

ε_{el} .....elastic strain

part 0-1 : linear portion of stress-strain diagram

$$E = E_1 = \frac{\sigma_1}{\varepsilon_1}$$

part 1-2 : plastic regime of the material

$$E = E_2 = \frac{\sigma_2}{\varepsilon_2 - \varepsilon_2'}$$

The line, describing unloading process of material is parallel with the linear part of stress-strain diagram. It is evident from this, that we are able to determine modulus of elasticity even if the loading is in the unlinear part of the stress-strain diagram. In this case we cannot calculate with whole measured deformation (because part of the deformation is permanent), but we have to unload the material, measure the deformation after unloading and calculate with the difference between deformation in loaded and unloaded stage of material.

Usually we don't know the exact shape of the stress-strain diagram of tested material before measuring and we don't know if the loading is in the linear or unlinear part of the stress-strain diagram. From this reason the process of the test consists of measuring in several loaded and unloaded stages.

15.2 Determination of Modulus of Elasticity

The method that have been used to measure modulus of elasticity are following: tension (or compression) test, bending test and natural frequency vibration test. The tension and bending test are based on the principle of Hook's law and they are called static methods. Measuring of the natural frequency of vibration gives dynamic modulus of elasticity. At our laboratory lesson we will determine static modulus of elasticity.

The static method is based on pulling or bending a sample of the material in an instrument which measures force and measuring the changes of the length. The change of the length is usually very small (tenths – hundredths of mm) and it is impossible to measure it by normal rules. For measuring are used special devices – extensometers. There are several types of extensometers as mechanical or electrical strain gauges.

15.2.1 Modulus of Elasticity in Tension

The test piece is mounted in the tensile testing machine which allows measurable forces to be applied. As extensometer is used mechanical strain gauge.

Tensile stress is counted from:

$$\sigma = \frac{F}{A} \quad [\text{MPa}]$$

where   F   is   applied force [N]
            A   cross sectional area [m^2]

Mechanical strain gauge (see Fig.:40) has two contacting point. One point is firm and the second is movable. The distance between points is called gauge (or original) length. between points is device which measures spacing between them. This device should be a dial where the small straight motion of the point is transmitted into bigger rotary motion of the hand. usually there are two hands – big hand showing divisions on main scale (centesimal or millesimal) and small hand, showing whole millimetres on auxiliary scale.

The change of the length is count as difference between gauge reading in the state of loading and gauge reading in unloaded state.

Fig.:40 Mechanical strain gauge
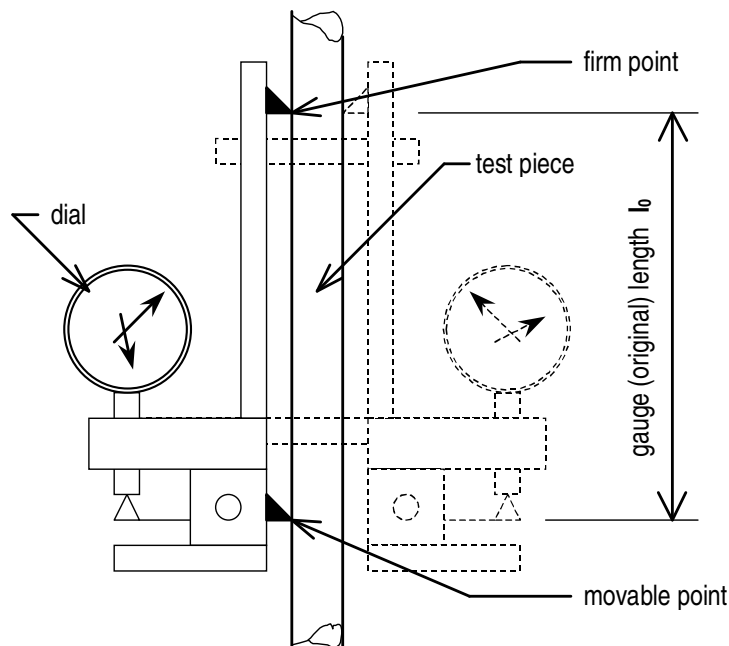
15.2.2 Modulus of Elasticity in Flexural Stress

The test piece is supported at either end and load (in a form of weights) is applied in the middle between the supports . As extensometer is used electrical resistance gauge.

Flexural stress is counted from:

$$\sigma = \frac{M}{W}$$

where   M   is   bending moment [N. mm^2]

W   section modulus [mm^3]

Some basic examples of loading and appropriate formulas for bending moment and section modulus are given in tab.3 (in chapter 3).

The principal of electrical resistance gauge is based on the fact that a change in electrical resistance is proportional to the strain, i.e. $\Delta R/R$ is function of ε.

Electrical resistance gauge consists of a thin wire grid, fixed on foil (usually paper). The gauge is bonded to the strained surface of measured material by a thin layer of a special glue. (see Fig.:41).

Fig.:41 Electrical resistance gauge
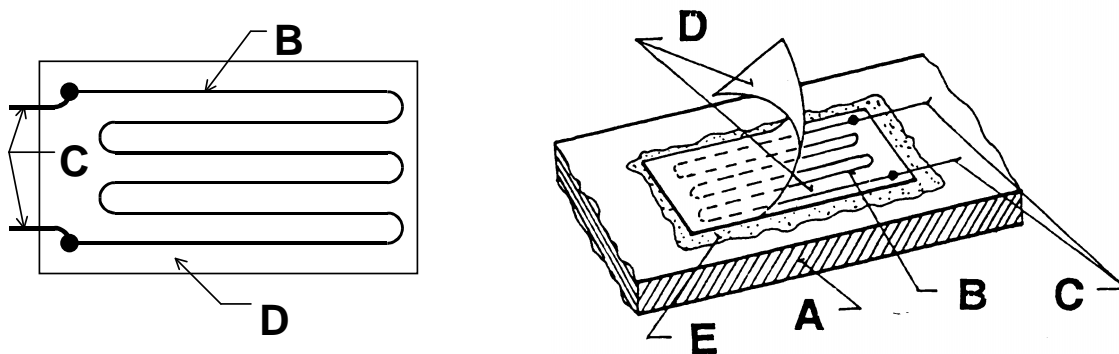
A   tested material   **D**   basic foil
B   wire grid   **E**   glue layer
C   connected leads

The change of resistance is proportional to strain according formula: $\dfrac{\Delta l}{l} = \dfrac{1}{K} \times \dfrac{\Delta R}{R}$

where   **l**   is   length of the wire

R   electrical resistance

K   gauge factor

The ideal strain gauge would change resistance only due to the deformations of the surface to which the gauge is attached. However, in real applications, temperature, material properties, the adhesive affect the detected resistance.

Internal influences (as materials properties of wire, type of used glue, size of the wire and the shape of the wire grid) are considered in gauge factor K.

The external influences (the changes of the temperature and moisture) have to be considered in a type of measuring circuit.

In order to measure strain with a bonded resistance strain gauge, it must be connected to an electric circuit that is capable of measuring the small changes in resistance corresponding to strain. Usually the Wheatstone bridge is used, this circuit is also well suited for temperature compensation.

Wheatstone bridge is a four arm, four-terminal resistance –measuring network.(see Fig.:42)

Fig.:42 Scheme of Wheatstone bridge
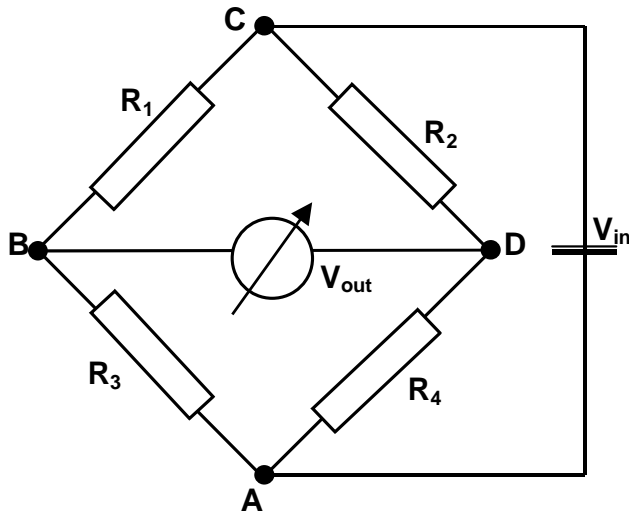
If the formula: $\dfrac{R_1}{R_2} = \dfrac{R_3}{R_4}$ is valid and voltage V_{in} is applied between points A and C, then the output between B and D will show no potential difference = the Wheatstone bridge is balanced (the galvanometer between B and D shows zero).

If R_1 is changed to some value which does not equal R_2, R_3, R_4, the bridge will become unbalanced and a voltage will exist at the output terminals.

Measuring scheme by apparatus TSA is given in Fig.:43. Resistance R_1 is a measuring gauge, bonded on tested material in the place of measured deformations. Resistance R_2 is compensating gauge. It is the same type of gauge as R_1, but bonded on non-loaded piece of material. It is used for temperature compensation, but not for strain measurement. The resistance changes due the temperature will have the same value for R_1 and R_2. This changes will cancel, because in the formula for balanced bridge resistances R_1 and R_2 are in ratio. Unbalance of the bridge will than caused only changes of R_1 from deformations.

Fig.:43 Measuring of modulus of elasticity by the electrical resistant gauge
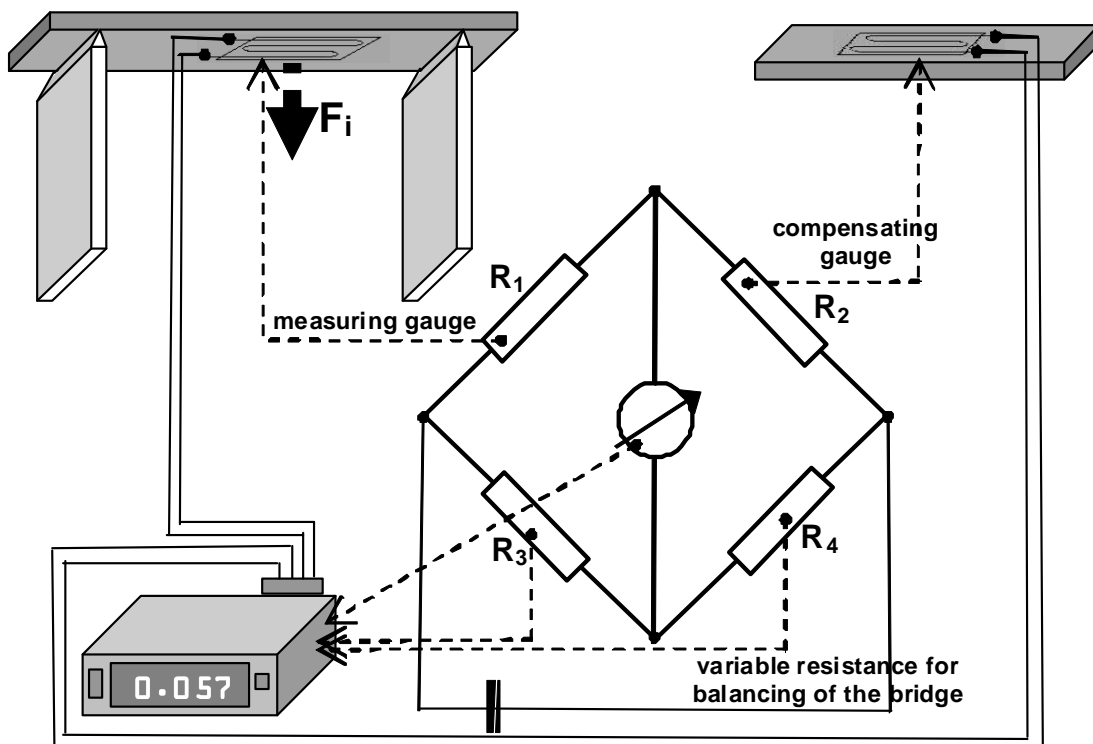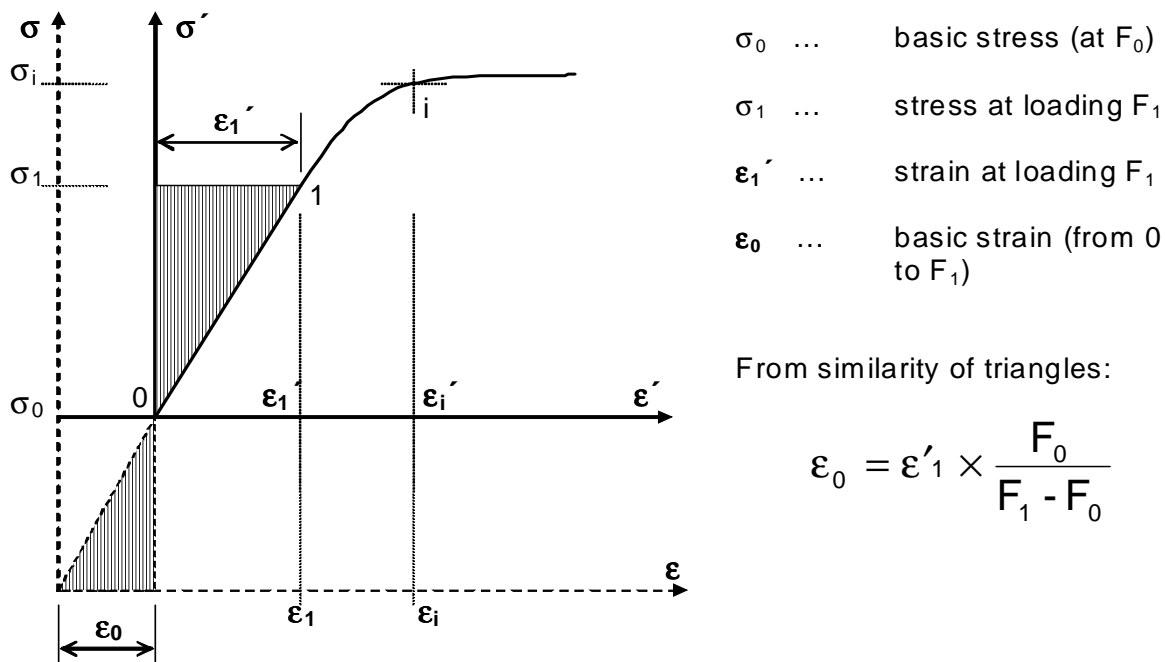
Resistances R_3 and R_4 are the components of tensometric apparatus. R_3 is adjustable and serves for nullification of gauge factor. Value of R_3 can be set up in advance. Resistance R_4 is a potentiometer (variable resistance), by it the bridge can be balanced again. The dial of the potentiometer has scale division straight in values of the strain.

15.2.3 Determining of the Basic Strain at Basic Loading

From several reasons it is impossible to measure stress and strain in the low loading range (close to zero loading). Mechanical and electrical machines have lost motions (motion is initiated, but you don't see any results for some time), or there is a risk of the specimen's falling-out from jaws.

If we determine initial modulus of elasticity (i.e. from zero loading to the appropriate loading), we cannot measure deformations straight from zero loading, we begin to measure from some basic loading F_0. We are able to determine basic part of deformations ε_0 (from zero to F_1) from linear part of the stress-strain diagram (see Fig.:44) by calculation or graphically from similarity of triangles.

Fig.:44 Extrapolation of stress-strain curve in the low loading range

σ_0 ... basic stress (at F_0)

σ_1 ... stress at loading F_1

$\varepsilon_1{'}$ ... strain at loading F_1

ε_0 ... basic strain (from 0 to F_1)

From similarity of triangles:

$$\varepsilon_0 = \varepsilon{'}_1 \times \frac{F_0}{F_1 - F_0}$$

Total strain at loading F_i is than :

$$\varepsilon_i = \varepsilon_0 + \varepsilon_i^{'}$$

where ε_i is   total strain from zero to appropriate loading F_i

   $\varepsilon_i{'}$ strain from F_1 to appropriate loading F_i

   ε_0 basic strain from zero to F_1
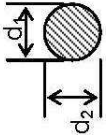
Tab.:36 Value of modulus of elasticity for some materials

Material	Modulus of elasticity [GPa]
aluminium and its alloys	65 - 73
ceramics	8 - 12
concrete	15 - 40
copper	125
diamond	1000
lightweight concrete	0,8 - 2
low alloy steels	200 - 210
polystyrene	3,2 - 3,5
silica glass	60 - 90
wood	11-16

📖 Vocabulary

balanced Wheatstone bridge	*Wheatstoneův můstek v rovnováze*
compensating gauge	*kompenzační tenzometr*
dial strain gauge	*číselníkový úchylkoměr*
elastic strain	*pružná (elastická) deformace*
electrical resistance strain gauge	*elektrický odporový tenzometr*
gauge factor	*konstanta tenzometru*
initial modulus of elasticity	*počáteční modul pružnosti*
lost motion	*mrtvý chod*
measuring gauge	*měřící tenzometr*
mechanical strain gauge	*mechanický tenzometr*
permanent strain	*trvalá (plastická) deformace*
slope	*směrnice*
stiffness	*tuhost*
strain	*deformace*
strain gauge	*tenzometr*
stress	*napětí*
stress-strain diagram	*pracovní diagram*
total strain	*celková deformace*
Wheatstone bridge	*Wheatstoneův můstek*

Fig.:45 Example of calculation of modulus of elasticity

MODULUS OF ELASTICITY IN FLEXURAL STRESS

Material : Fibre-glass polyester laminate

Cross sectional size: $d_1 = 16,4$ mm $d_2 = 16,4$ mm

Source of loading : Set of weights

Section modulus :
$$W = \frac{\pi}{32} d^3 = 441,09 \ mm^3$$

Measuring apparatus : Electrical resistance strain gauge, gauge factor $k = 1,95$

Basic strain (from 0 to F_0) :
$$\varepsilon_0 = \varepsilon_1' \frac{F_0}{F_1 - F_0} = 0,0525$$

Loading scheme :

$$M_i = \frac{F_i \times l}{4}$$

$$l = 760 \ [mm]$$

$$E = \frac{\sigma_i}{(\varepsilon_i' + \varepsilon_0).10^{-3}} \quad [MPa]$$

Loading F_i	Reading of apparatus		Strain during unloading from F_i to F_0		mean ε_i'	Total strain $\varepsilon_i' + \varepsilon_0$	Bending moment M_i	Stress $\sigma_i = \dfrac{M_i}{W}$	Modulus of elasticity $E = \dfrac{\sigma_i}{(\varepsilon_i + \varepsilon_0).10^{-3}}$
	1.	2.	1.	2.					
[N]	‰		‰		‰	‰	[N. mm]	[MPa]	[MPa]
F_1 = 10	17,768	17,730							
F_0 = 5	17,715	17,678	0,053	0,052	0,0525	0,105	1900	4,307	41 019
F_2 = 15	17,819	17,780							
F_0 = 5	17,716	17,676	0,103	0,104	0,1035	0,156	2850	6,461	41 417
F_3 = 20	17,870	17,831							
F_0 = 5	17,720	17,678	0,150	0,153	0,1515	0,204	3800	8,615	42 230

Mean value of modulus of elasticity : 41,5 GPa